CAMBRIDGE ASSESSMENT

STEP Solutions 2008

Mathematics

STEP 9465, 9470, 9475

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STEP Mathematics II 2008: Solutions

1 (i) Given $(x_{n+1}, y_{n+1}) = (x_n^2 - y_n^2 + 1,\ 2x_n y_n + 1)$, it is easier to remove the subscripts and set $x^2 - y^2 + 1 = x$ and $2xy + 1 = y$. Then, identifying the y's (or x's) in each case, gives $y^2 = x^2 - x + 1$ and $y = \dfrac{1}{1-2x}$. Eliminating the y's leads to a polynomial equation in x; namely, $4x^4 - 8x^3 + 9x^2 - 5x = 0$.

Noting the obvious factor of x, and then finding a second linear factor (e.g. by the factor theorem) leads to $x(x-1)(4x^2 - 4x + 5) = 0$. Here, the quadratic factor has no real roots, since the discriminant, $\Delta = 4^2 - 4.4.5 = -64 < 0$. [Alternatively, one could note that $4x^2 - 4x + 5 \equiv (2x-1)^2 + 4 > 0\ \forall x$.]

The two values of x, and the corresponding values of y, gained by substituting these x's into $y = \dfrac{1}{1-2x}$, are then $(x, y) = (0, 1)$ and $(1, -1)$

(ii) Now $(x_1, y_1) = (-1, 1) \Rightarrow (x_2, y_2) = (a, b)$ and $(x_3, y_3) = (a^2 - b^2 + a, 2ab + b + 2)$. Setting both $a^2 - b^2 + a = -1$ and $2ab + b + 2 = 1$, so that the third term is equal to the first, and identifying the b's in each case, gives $b^2 = a^2 + a + 1$ and $b = \dfrac{-1}{1+2a}$.

One could go about this the long way, as before. However, it can be noted that the algebra is the same as in (i), but with $a = -x$ and $b = -y$. Either way, we obtain the two possible solution-pairs: $(a, b) = (0, -1)$ and $(-1, 1)$.

However, upon checking, the solution $(-1, 1)$ actually gives rise to a constant sequence (and remember that the working only required the third term to be the same as the first, which doesn't preclude the possibility that it is also the same as the second term!), so we find that there is in fact just the one solution: $(a, b) = (0, -1)$.

2 The correct partial fraction form for the given algebraic fraction is

$$\frac{1+x}{(1-x)^2(1+x^2)} \equiv \frac{A}{1-x} + \frac{B}{(1-x)^2} + \frac{Cx+D}{1+x^2},$$

although these *can* also be put together in other correct ways that don't materially hinder the progress of the solution. The standard procedure now is to multiply throughout by the denominator of the LHS and compare coefficients or substitute in suitable values: which leads to $A = \frac{1}{2}$, $B = 1$, $C = \frac{1}{2}$ and $D = -\frac{1}{2}$.

In order to apply the binomial theorem to these separate fractions, we now use index notation to turn

$$\frac{1+x}{(1-x)^2(1+x^2)} \equiv A(1-x)^{-1} + B(1-x)^{-2} + Cx(1+x^2)^{-1} + D(1+x^2)^{-1}$$

into the infinite series

$$\tfrac{1}{2}\sum_{n=0}^{\infty} x^n + \sum_{n=0}^{\infty}(n+1)x^n + \tfrac{1}{2}\sum_{n=0}^{\infty}(-1)^n x^{2n+1} - \tfrac{1}{2}\sum_{n=0}^{\infty}(-1)^n x^{2n}.$$

It should be clear at this point that the last two of these series have odd/even powers only, with alternating signs playing an extra part. The consequence of all this is that we need to examine cases for n modulo 4; i.e. depending upon whether n leaves a remainder of 0, 1, 2 or 3 when divided by 4.

For $n \equiv 0 \pmod 4$, the coefft. of x^n is $\frac{1}{2} + n + 1 + 0 - \frac{1}{2} = n + 1$;

A1 for $n \equiv 1 \pmod 4$, coefft. of x^n is $\frac{1}{2} + n + 1 + \frac{1}{2} - 0 = n + 2$;

A1 for $n \equiv 2 \pmod 4$, coefft. of x^n is $\frac{1}{2} + n + 1 + 0 + \frac{1}{2} = n + 2$;

A1 for $n \equiv 3 \pmod 4$, coefft. of x^n is $\frac{1}{2} + n + 1 - \frac{1}{2} + 0 = n + 1$.

For the very final part of the question, we note that $\frac{11000}{8181} = \frac{1.1}{0.9^2 \times 1.01}$, is a cancelled form of our original expression, with $x = 0.1$. (N.B. $|x| < 1$ assures the convergence of the infinite series forms). Substituting this value of x into

$$1 + 3x + 4x^2 + 4x^3 + 5x^4 + 7x^5 + 8x^6 + 8x^7 + 9x^8 + \ldots$$

then gives 1.344 578 90 to 8dp.

3 (i) Setting $\frac{dy}{dx} = 81x^2 - 54x = 0$ for TPs gives $(0, 4)$ and $\left(\frac{2}{3}, 0\right)$. You really ought to know the shape of such a ("positive") cubic, and it is customary to find the crossing-points on the axes: $x = 0$ gives $y = 4$, and $y = 0$ leads to $x = -1$ and $x = \frac{2}{3}$ (twice). [If you have been paying attention, this latter zero for y should come as no surprise!] The graph now shows that, for all $x \geq 0$, $y \geq 0$; which leads to the required result – $x^2(1 - x) \leq \frac{4}{27}$ – with just a little bit of re-arrangement.

In order to prove the result by contradiction (*reduction ad absurdum*), we first assume that all three numbers exceed $\frac{4}{27}$. Then their product

$$bc(1 - a)ca(1 - b)ab(1 - c) > (\tfrac{4}{27})^3.$$

However, this product can be re-written in the form

$$a^2(1 - a).\ b^2(1 - b.)\ c^2(1 - c),$$

and the previous result guarantees that $x^2(1 - x) \leq \frac{4}{27}$ for each of a , b , c, from which it follows that

$$a^2(1 - a).\ b^2(1 - b.)\ c^2(1 - c) \leq (\tfrac{4}{27})^3,$$

which is the required contradiction. Hence, at least one of the three numbers $bc(1 - a)$, $ca(1 - b)$, $ab(1 - c)$ is less than, or equal to, $\frac{4}{27}$.

(ii) Drawing the graph of $y = x - x^2$ (there are, of course, other suitable choices, such as $y = (2x - 1)^2$ for example) and showing that it has a maximum at $(\frac{1}{2}, \frac{1}{4})$ gives

$$x(1 - x) \leq \tfrac{1}{4} \text{ for all } x.$$

The assumption that $p(1 - q)$, $q(1 - p) > \frac{1}{4} \Rightarrow p(1 - p).q(1 - q) > (\frac{1}{4})^2$.

However, we know that $x(1 - x) \leq \frac{1}{4}$ for each of p and q, and this gives us that

$$p(1 - p).q(1 - q) \leq (\tfrac{1}{4})^2.$$

Hence, by contradiction, at least one of $p(1 - q)$, $q(1 - p) \leq \frac{1}{4}$.

4 Differentiating implicitly gives $2\left(x + y\dfrac{\mathrm{d}y}{\mathrm{d}x} + ax\dfrac{\mathrm{d}y}{\mathrm{d}x} + ay\right) = 0$, from which it follows that

$\dfrac{\mathrm{d}y}{\mathrm{d}x} = -\dfrac{x+ay}{ax+y}$ and hence the gradient of the normal is $\dfrac{ax+y}{x+ay}$.

Using $\tan(A - B)$ on this and $\dfrac{y}{x}$ gives $\tan\theta = \left|\dfrac{\dfrac{y}{x} - \dfrac{ax+y}{x+ay}}{1 + \dfrac{y}{x}\times\dfrac{ax+y}{x+ay}}\right| = \left|\dfrac{xy + ay^2 - ax^2 - xy}{x^2 + axy + axy + y^2}\right|$.

However, we know that $x^2 + y^2 + 2axy = 1$ from the curve's eqn., and so

$$\tan\theta = a\left|y^2 - x^2\right|.$$

(i) Differentiating this w.r.t. x then gives $\sec^2\theta\,\dfrac{\mathrm{d}\theta}{\mathrm{d}x} = a\left(2y\dfrac{\mathrm{d}y}{\mathrm{d}x} - 2x\right)$. Equating this to zero and using $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -\dfrac{x+ay}{ax+y}$ from earlier then leads to $a(x^2 + y^2) + 2xy = 0$.

(ii) Adding $x^2 + y^2 + 2axy = 1$ and $a(x^2 + y^2) + 2xy = 0$ gives $(1 + a)(x + y)^2 = 1$.

(iii) However, subtracting these two eqns. instead gives $(1 - a)(y - x)^2 = 1$, and multiplying these two last results together yields $(1 - a^2)(y^2 - x^2)^2 = 1$.

Finally, using $\tan\theta = a\left|y^2 - x^2\right| \Rightarrow (y^2 - x^2)^2 = \dfrac{1}{a^2}\tan^2\theta$, and substituting this into the last result of (iii) then gives the required result: $\tan\theta = \dfrac{a}{\sqrt{1-a^2}}$. All that remains is to justify taking the positive square root, since $\tan\theta$ is | something |, which is necessarily non-negative.

5 Using a well-known double-angle formula gives $\displaystyle\int_0^{\pi/2} \frac{\sin 2x}{1+\sin^2 x}\,\mathrm{d}x = \int_0^{\pi/2} \frac{2\sin x\cos x}{1+\sin^2 x}\,\mathrm{d}x$, and this should suggest an obvious substitution: letting $s = \sin x$ turns this into the integral

$$\int_0^1 \frac{2s}{1+s^2}\,\mathrm{d}s .$$

This is just a standard log. integral (the numerator being the derivative of the denominator), leading to the answer $\ln 2$.

Alternatively, one could use the identity $\sin^2 x \equiv \tfrac{1}{2} - \tfrac{1}{2}\cos 2x$ to end up with

$$\int_0^{\pi/2} \frac{2\sin 2x}{3 - \cos 2x}\,\mathrm{d}x.$$

This, again, gives a log. integral, but without the substitution.

A suitable substitution for the second integral is $c = \cos x$, which leads to $\displaystyle\int_0^1 \frac{1}{2-c^2}\,\mathrm{d}c$.

Now you can either attack this using partial fractions, or you could look up what is a fairly standard result in your formula booklet. In each case, you get (after a bit of careful log and surd work) $\frac{1}{\sqrt{2}}\ln\left(1+\sqrt{2}\right)$.

Now $(1+\sqrt{2})^5 = 1 + 5\sqrt{2} + 20 + 20\sqrt{2} + 20 + 4\sqrt{2} = 41 + 29\sqrt{2}$ (using the binomial theorem, for instance), and

$$41 + 29\sqrt{2} < 99 \Leftrightarrow 29\sqrt{2} < 58 \Leftrightarrow \sqrt{2} < 2,$$

which is obviously the case. Also, $1.96 < 2 \Rightarrow 1.4 < \sqrt{2}$. Thereafter, an argument such as

$$2^{1.4} > 1 + \sqrt{2} \Leftrightarrow 2^7 > (1+\sqrt{2})^5 \Leftrightarrow 128 > 41 + 29\sqrt{2}$$
$$\Leftrightarrow \quad 87 > 29\sqrt{2} \quad \Leftrightarrow \quad 3 > \sqrt{2}$$

from which it follows that $2^{\sqrt{2}} > 2^{7/5} > 1+\sqrt{2}$.

Taking logs in this result then gives $\sqrt{2}\ \ln 2 > \ln(1+\sqrt{2}) \Rightarrow \ln 2 > \frac{1}{\sqrt{2}}\ln\left(1+\sqrt{2}\right)$; and

$$\int_0^{\pi/2} \frac{\sin 2x}{1+\sin^2 x}\,\mathrm{d}x > \int_0^{\pi/2} \frac{\sin x}{1+\sin^2 x}\,\mathrm{d}x.$$

6 (i) Firstly, $\cos x$ has period $2\pi \Rightarrow \cos(2x)$ has period π;

and $\sin x$ has period $2\pi \Rightarrow \sin\left(\frac{3x}{2}\right)$ has period $\frac{4}{3}\pi$.

Then $f(x) = \cos\left(2x+\frac{\pi}{3}\right) + \sin\left(\frac{3x}{2}-\frac{\pi}{4}\right)$ has period $4\pi = \mathrm{lcm}(\pi, \frac{4}{3}\pi)$.

(ii) Any approach here is going to require the use of some trig. identity work. The most straightforward is to note that $\cos\left(\frac{\pi}{2}+\theta\right) = -\sin\theta$ so that $f(x) = 0$ reduces to

$\cos\left(2x+\frac{\pi}{3}\right) = \cos\left(\frac{3x}{2}+\frac{\pi}{4}\right)$, from which it follows that $2x+\frac{\pi}{3} = 2n\pi \pm \left(\frac{3x}{2}+\frac{\pi}{4}\right)$

where n is an integer, using the symmetric and periodic properties of the cosine curve. Taking suitable values of n, so that x is in the required interval, leads to the answers

$x = -\frac{31\pi}{42}$ (from $n = -1$, with the – sign), $x = -\frac{\pi}{6}$ ($n = 0$, with both + and – signs),

$x = \frac{17\pi}{42}$ ($n = 1$, – sign) and $x = \frac{41\pi}{42}$ ($n = 2$, – sign).

Since $x = -\frac{\pi}{6}$ is a repeated root (occurring twice in the above list), the curve of $y = f(x)$ touches the x-axis at this point.

For those who are aware of the results that appear in all the formula books, but which seem to be on the edge of the various syllabuses, that I know by the title of the *Sum-and-Product Formulae*, such as $\cos A + \cos B \equiv 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$, there is a second straightforward approach available here. For example, noting that

$\cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta$ gives $\cos\left(2x+\frac{\pi}{3}\right) + \cos\left(\frac{3\pi}{4}-\frac{3x}{2}\right) = 0$ which (from the above identity) then gives $2\cos\left(\frac{x}{4}+\frac{13\pi}{24}\right)\cos\left(\frac{7x}{4}-\frac{5\pi}{24}\right) = 0$, and setting each of these two cosine terms equal to zero, in turn, yields the same values of x as before, including the repeat.

(iii) The key observation here is that $y = 2$ if and only if *both* $\cos\left(2x+\frac{\pi}{3}\right) = 1$ *and* $\sin\left(\frac{3x}{2}-\frac{\pi}{4}\right) = 1$, simultaneously. So we must solve

$\cos\left(2x+\frac{\pi}{3}\right) = 1 \Rightarrow 2x+\frac{\pi}{3} = 0, 2\pi, 4\pi, \ldots$, giving $x = \frac{5\pi}{6}, \frac{11\pi}{6}, \ldots$; and

$\sin\left(\frac{3x}{2}-\frac{\pi}{4}\right) = 1 \Rightarrow \frac{3x}{2}-\frac{\pi}{4} = \frac{\pi}{2}, \frac{5\pi}{2}, \ldots$, giving $x = \frac{\pi}{2}, \frac{11\pi}{6}, \ldots$.

Both equations are satisfied when $x = \frac{11\pi}{6}$, and this is the required answer.

7 (i) Differentiating $y = u\sqrt{1+x^2}$ gives $\frac{dy}{dx} = u.\frac{x}{\sqrt{1+x^2}} + \sqrt{1+x^2}.\frac{du}{dx}$; so that

$\frac{1}{y}\frac{dy}{dx} = xy + \frac{x}{1+x^2}$ becomes $\frac{1}{u\sqrt{1+x^2}}\left\{\frac{ux}{\sqrt{1+x^2}} + \sqrt{1+x^2}.\frac{du}{dx}\right\} = xu\sqrt{1+x^2} + \frac{x}{1+x^2}$.

Simplifying and cancelling the common term on both sides leads to

$$\frac{1}{u}.\frac{du}{dx} = xu\sqrt{1+x^2}.$$

This is a standard form for a first-order differential equation, involving the separation of variables and integration:

$$\int \frac{1}{u^2}.du = \int x\sqrt{1+x^2}\,dx \Rightarrow -\frac{1}{u} = \frac{1}{3}\left(1+x^2\right)^{3/2} \ (+C).$$

Using $x = 0$, $y = 1$ ($u = 1$) to find C leads to the final answer, $y = \frac{3\sqrt{1+x^2}}{4-\left(1+x^2\right)^{3/2}}$.

(ii) The key here is to choose the appropriate function of x. If you have really got a feel for what has happened in the previous bit of the question, then this isn't too demanding. If you haven't really grasped fully what's going on then you may well need to try one or two possibilities first. The product that needs to be identified here is $y = u\left(1+x^3\right)^{1/3}$. Once you have found this, the process of (i) pretty much repeats itself.

$\frac{dy}{dx} = u.x^2\left(1+x^3\right)^{-2/3} + \left(1+x^3\right)^{1/3}\frac{du}{dx}$ means that $\frac{1}{y}\frac{dy}{dx} = x^2y + \frac{x^2}{1+x^3}$ becomes

$$\frac{1}{u}.\frac{du}{dx} = x^2u\left(1+x^3\right)^{1/3}.$$

Separating variables and integrating:

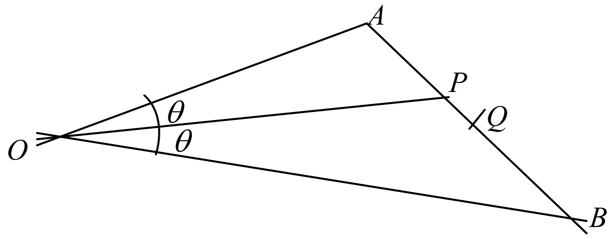
$$\int \frac{1}{u^2}.du = \int x^2\left(1+x^3\right)^{1/3}dx = -\frac{1}{u} = \frac{1}{4}\left(1+x^3\right)^{4/3} \ (+C);$$

and $x = 0$, $y = 1$ ($u = 1$) gives C and the answer $y = \dfrac{4\left(1+x^3\right)^{1/3}}{5-\left(1+x^3\right)^{4/3}}$.

(iii) Note that the question didn't actually require you to simplify the two answers in (i) and (ii), but doing so certainly enables you to have a better idea as to how to generalise the results:

$$y = \frac{(n+1)\left(1+x^n\right)^{1/n}}{(n+2)-\left(1+x^n\right)^{1+1/n}} .$$

8 It is never a bad idea to start this sort of question with a reasonably accurate diagram … something along the lines of



The first result is an example of what is known as the *Ratio Theorem*:

$$AP : PB = 1 - \lambda : \lambda \Rightarrow \mathbf{p} = \lambda\mathbf{a} + (1-\lambda)\mathbf{b} .$$

Alternatively, it can be deduced from the standard approach to the vector equation of a straight line, via $\mathbf{r} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a})$.

Using the scalar product twice then gives

$$\mathbf{a} \bullet \mathbf{p} = \lambda a^2 + (1-\lambda)(\mathbf{a} \bullet \mathbf{b}) \text{ and } \mathbf{b} \bullet \mathbf{p} = \lambda(\mathbf{a} \bullet \mathbf{b}) + (1-\lambda)\, b^2 .$$

Equating these two expressions for $\cos\theta$, $\dfrac{\mathbf{a}\bullet\mathbf{p}}{ap} = \dfrac{\mathbf{b}\bullet\mathbf{p}}{bp}$, re-arranging and collecting up like terms, then gives $ab\{\lambda(a+b) - b\} = \mathbf{a} \bullet \mathbf{b}\ \{\lambda(a+b) - b\}$. There are two possible consequences to this statement, and *both* of them should be considered. Either $ab = \mathbf{a} \bullet \mathbf{b}$, which gives $\cos 2\theta = 1$, $\theta = 0$, $A = B$ and violates the non-collinearity of O, A & B; or the bracketed factor on each side is zero, which gives

$$\lambda = \frac{b}{a+b} .$$

However, if you know the *Angle Bisector Theorem*, the working is short-circuited quite dramatically:

$$\frac{AP}{PB} = \frac{OA}{OB} \Rightarrow \frac{(1-\lambda)(AB)}{\lambda(AB)} = \frac{a}{b} \Rightarrow b - b\lambda = a\lambda \Rightarrow \lambda = \frac{b}{a+b} .$$

Next, $AQ : QB = \lambda : 1 - \lambda \Rightarrow \mathbf{q} = (1-\lambda)\mathbf{a} + \lambda\mathbf{b}$.
Then

$$OQ^2 = \mathbf{q} \bullet \mathbf{q} = (1-\lambda)^2 a^2 + \lambda^2 b^2 + 2\lambda(1-\lambda)\, \mathbf{a} \bullet \mathbf{b}$$

and $OP^2 = \mathbf{p} \bullet \mathbf{p} = (1-\lambda)^2 b^2 + \lambda^2 a^2 + 2\lambda(1-\lambda)\, \mathbf{a} \bullet \mathbf{b}$.

[N.B. This working can also be done by the *Cosine Rule*.]

Subtracting:

$$OQ^2 - OP^2 = (b^2 - a^2)\,[\lambda^2 - (1-\lambda)^2] = (b^2 - a^2)\,(2\lambda - 1)$$

and, substituting λ in terms of a and b into this expression, gives the required answer

$$= (b-a)(b+a) \times \frac{b-a}{b+a} = (b-a)^2.$$

9 (i) Using a modified version of the trajectory equation (which you are encouraged to have learnt), $y = h + x\tan\alpha - \frac{gx^2}{2u^2}\sec^2\alpha$, and substituting in $g = 10$ and $u = 40$ gives

$$y = h + x\tan\alpha - \frac{gx^2}{320}\sec^2\alpha .$$

Setting $x = 20$ and $y = 0$ into this trajectory equation and using one of the well-known *Pythagorean* trig. identities ($\sec^2\alpha = 1 + \tan^2\alpha$) leads to the quadratic equation

$$5t^2 - 80t - (4h - 5) = 0$$

in $t = \tan\alpha$.
[Note that you could have substituted $x = 20$ and $y = -h$ into the unmodified trajectory equation and still got the same result here.]
Solving, using the quadratic formula, and simplifying then gives

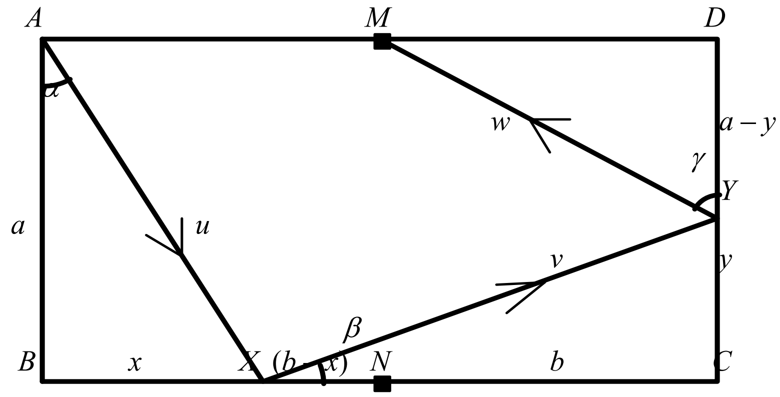
$$\tan\alpha = 8 \pm \sqrt{63 + \tfrac{4}{5}h} .$$

We reject $\tan\alpha = 8 + \sqrt{63 + \tfrac{4}{5}h}$, since this gives a very high angle of projection and hence a much greater time for the ball to arrive at the stumps. Now, since α is small, $\cos\alpha \approx 1$, and the time of flight $= \frac{x}{u\cos\alpha} = \frac{1}{2\cos\alpha} \approx \frac{1}{2}$.

(ii) $h > \frac{5}{4}$ for $\tan\alpha = 8 - \sqrt{64 + \varepsilon}\ ) < 0$.

(iii) Now $h = 2.5 \Rightarrow \tan\alpha = 8 - \sqrt{64 + 1} = 8 - 8\left(1 + \frac{1}{64}\right)^{1/2}$. The *Binomial Theorem* then allows us to expand the bracket, and it seems reasonable to take just the first term past the 1: $\tan\alpha = 8 - 8\left(1 + \frac{1}{128} + ...\right)$, so that $\tan\alpha \approx -\frac{1}{16}$. [We can ignore the minus sign, since this just tells us that the projection is *below* the horizontal.]
Using $\tan\alpha \approx \alpha$ for small-angles, and converting from radians into degrees using the conversion factor $180/\pi \approx 57.3$ then gives $\alpha \approx 3.6^\circ$.

10 On this sort of question, a good, clear diagram is almost essential, even when it is not asked-for.



(i) The two fundamental principles involved in collisions are the *Conservation of Linear Momentum* (*CLM*) and *Newton's Experimental Law of Restitution* (*NEL* or *NLR*).
For the collision at *X*, applying *CLM* || *BC* $\Rightarrow mu\sin\alpha = mv\cos\beta$

.. *NEL* $\Rightarrow eu\cos\alpha = v\sin\beta$

Dividing these two gives $\tan\beta = e\cot\alpha$ or $\tan\alpha\tan\beta = e$.
At Y, "similarly", we have $\tan\beta\tan\gamma = e$. Hence $\alpha = \gamma$ (since all angles are acute).

(ii) A good approach here is to use similar Δs, and a bit of sensible labelling is in order (see the diagram). Let $BX = x$ ($XN = b - x$) and $CY = y$ ($DY = a - y$). Then

$$\tan\alpha = \frac{x}{a}, \quad \tan\beta = \frac{y}{2b - x}, \quad \tan\gamma = \frac{b}{a - y}.$$

Using $\alpha = \gamma$ to find (e.g) y in terms of $a, b, x \Rightarrow ax - xy = ab \Rightarrow y = \dfrac{a(x-b)}{x}$.

Next, we use the result $\tan\alpha\tan\beta = e$ from earlier to get x in terms of a and b:

$$\frac{x}{a} \times \frac{a(x-b)/x}{2b - x} = e \Rightarrow x - b = 2be - ex \Rightarrow x = \frac{b(1+2e)}{1+e},$$

from which it follows that $\tan\alpha = \dfrac{b(1+2e)}{a(1+e)}$.

At this stage, some sort of inequality argument needs to be considered, and a couple of obvious approaches might occur to you.

I $\tan\alpha = \dfrac{b(1+2e)}{a(1+e)} = \dfrac{b}{a} + \dfrac{be}{a(1+e)} > \dfrac{b}{a}$ and $\tan\alpha = \dfrac{b(1+2e)}{a(1+e)} = \dfrac{2b}{a} - \dfrac{be}{a(1+e)} < \dfrac{2b}{a}$

give $\dfrac{b}{a} < \tan\alpha < \dfrac{2b}{a}$; and the shot is possible, with the ball striking BC between N and C, whatever the value of e.

II As $e \to 0$, $\tan\alpha \to \dfrac{b}{a}+$ and as $e \to 1$, $\tan\alpha \to \dfrac{3b}{2a}-$, so that

$\dfrac{b}{a} < \tan\alpha < \dfrac{3b}{2a}$; and the shot is possible, with the ball striking BC between N and the midpoint of NC, whatever the value of e.

(iii) There are two possible approaches to this final part. The first, much longer version, involves squaring and adding the eqns. for the collision at X, and then again at Y, to get

$$v^2 = u^2(\sin^2\alpha + e^2\cos^2\alpha) \quad \text{and} \quad w^2 = v^2(\sin^2\beta + e^2\cos^2\beta).$$

Now, noting that the initial KE $= \frac{1}{2}mu^2$ and the final KE $= \frac{1}{2}mw^2$, the fraction of KE lost is

$$\frac{\frac{1}{2}mu^2 - \frac{1}{2}mw^2}{\frac{1}{2}mu^2} = 1 - \frac{w^2}{u^2} = 1 - (\sin^2\alpha + e^2\cos^2\alpha)(\sin^2\beta + e^2\cos^2\beta)$$

$$= 1 - \frac{\tan^2\alpha + e^2}{\sec^2\alpha} \times \frac{\tan^2\beta + e^2}{\sec^2\beta}.$$

From here, we use $\tan\alpha\tan\beta = e$ and $\sec^2\alpha = 1 + \tan^2\alpha$ to get

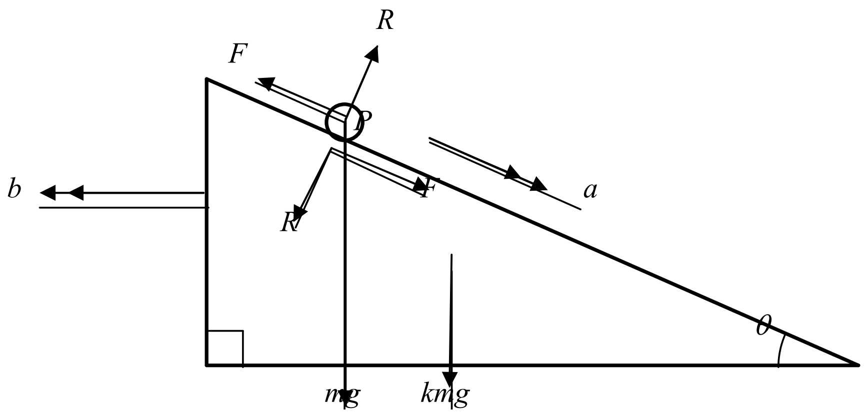
$$1 - \frac{t^2 + e^2}{1 + t^2} \times \frac{e^2/t^2 + e^2}{1 + e^2/t^2} = 1 - \frac{t^2 + e^2}{1 + t^2} \times \frac{e^2\left(1 + t^2\right)/t^2}{\left(t^2 + e^2\right)/t^2} = 1 - e^2, \text{ as required.}$$

However, it is very much quicker to note the following:

At X, the ↑-component of the ball's velocity becomes $e \times$ initial ↑-component ,
and
at Y, the →-component of the ball's velocity becomes $e \times$ initial →-component.
Hence its final velocity is eu and the fraction of the KE lost is then

$$\frac{\frac{1}{2}mu^2 - \frac{1}{2}me^2u^2}{\frac{1}{2}mu^2} = 1 - e^2.$$

11



Once again, a good, clear diagram is an important starting-point, and the above diagram shows the relevant forces – labelled using standard notations – along with the accelerations of P down the sloping surface of the wedge (a) and the wedge itself along the plane (b).

(i) Noting the acceleration components of P are $a\cos\theta - b$ ($\rightarrow$) and $a\sin\theta$ ($\downarrow$), we employ *Newton's Second Law* as follows:

N2L → for P $\quad m(a\cos\theta - b) = R\sin\theta - F\cos\theta$

N2L ↓ for P $\quad ma\sin\theta = mg - F\sin\theta - R\cos\theta$

N2L ← for wedge $\quad kmb = R\sin\theta - F\cos\theta$

From which it follows that $a\cos\theta - b = kb \Rightarrow b = \dfrac{a\cos\theta}{k+1}$.

Alternatively, one could use *N2L* to note *P*'s → accln. component and also the wedge's accln. ←, but instead use

CLM ↔ $\quad kmbt = m(a\cos\theta - b)t$ (where t = time from release)

and this again leads to the above result for b.

Now, for P to move at 45° to the horizontal, $a\cos\theta - b = a\sin\theta$. Then

$$b = a(\cos\theta - \sin\theta) = \frac{a\cos\theta}{k+1}$$

$\Rightarrow (k+1)(\cos\theta - \sin\theta) = \cos\theta \Rightarrow k + 1 - (k+1)\tan\theta = 1$ and $\tan\theta = \dfrac{k}{k+1}$.

When $k = 3$, $\tan\theta = \frac{3}{4}$, $\sin\theta = \frac{3}{5}$, $\cos\theta = \frac{4}{5}$ and $b = \frac{1}{5}a$.

Substituting these into the first two equations of motion from (i), along with the use of the *Friction Law (in motion)*, which assumes that $F = \mu R$, gives

$$m(\tfrac{4}{5}a - b) = \tfrac{3}{5}R - \tfrac{4}{5}F \quad \text{or} \quad 3R - 4F = m(4a - 5b) = 3ma \Rightarrow R(3 - 4\mu) = 3ma$$

and

$$\tfrac{3}{5}ma = mg - \tfrac{3}{5}F - \tfrac{4}{5}R \quad \text{or} \quad 4R + 3F = m(5g - 3a) \Rightarrow R(4 + 3\mu) = 5mg - 3ma.$$

Dividing, or equating for R:

$$\frac{4+3\mu}{3-4\mu} = \frac{5g-3a}{3a} \Rightarrow (12 + 9\mu)a = 5(3 - 4\mu)g - (9 - 12\mu)a \Rightarrow a = \frac{5(3-4\mu)g}{3(7-\mu)}.$$

(ii) Finally, if $\tan\theta \le \mu$, then both P and the wedge remain stationary. So, technically, the answer is "nothing".

12 Clearly, $X \in \{0, 1, 2, 3\}$ and working out the corresponding probabilities is a good thing to do at some point in this question (although it can, of course, be done numerically later when a value for p has been found).

$p(X = 0) = (1 - p)(1 - \tfrac{1}{3}p)(1 - p^2)$

$p(X = 1) = p(1 - \tfrac{1}{3}p)(1 - p^2) + (1 - p)\tfrac{1}{3}p(1 - p^2) + (1 - p)(1 - \tfrac{1}{3}p)p^2$

$\quad = p(1 - p)(\tfrac{4}{3} + \tfrac{5}{3}p - p^2)$

$p(X = 2) = p.\tfrac{1}{3}p(1 - p^2) + p(1 - \tfrac{1}{3}p)p^2 + (1 - p)\tfrac{1}{3}p.p^2$

$\quad = \tfrac{1}{3}p^2(1 + 4p - 3p^2)$

$p(X = 3) = \tfrac{1}{3}p^4$

[Of course, one of these could be deduced on a (1 – the sum of the rest) basis, but that can always be left as useful check on the correctness of your working, if you so wish.]

Then $E(X) = \sum x.p(x) = 0 + p(1 - p)(\tfrac{4}{3} + \tfrac{5}{3}p - p^2) + \tfrac{2}{3}p^2(1 + 4p - 3p^2) + p^4$

$\quad = \tfrac{4}{3}p + p^2$

Alternatively, if you have done a little bit of expectation algebra, it is clear that

$$E(X) = \sum E(X_i) = p + \tfrac{1}{3}p + p^2 = \tfrac{4}{3}p + p^2 .$$

Equating this to $\tfrac{4}{3} \Rightarrow 0 = 3p^2 + 4p - 4 \Rightarrow 0 = (3p - 2)(p + 2)$, and since $0 < p < 1$ it follows that $p = \tfrac{2}{3}$.

In the final part, you will need either (p_0 and p_1) or (p_2 and p_3):

$$p_0 = \tfrac{35}{243} \text{ and } p_1 = \tfrac{108}{243} \text{ or } \quad p_2 = \tfrac{84}{243} \text{ and } p_3 = \tfrac{16}{243} .$$

Next, a careful statement of cases is important (with, I hope, obvious notation):

p(correct pronouncement) = p(G and ≥ 2 judges say G) + p(NG and ≤ 1 judges say G)

$$= t \,.\, \frac{100}{243} + (1 - t)\,.\, \frac{143}{243} = \frac{143 - 43t}{243}$$

Equating this to $\tfrac{1}{2}$ and solving for $t \Rightarrow 243 = 286 - 86t \Rightarrow 86t = 43 \Rightarrow t = \tfrac{1}{2}$.

Alternatively, let p(King pronounces guilty) = q .

Then "King correct" = "King pronounces guilty and defendant *is* guilty"

or "King pronounces not guilty and defendant *is* not guilty"

so that p(King correct) = $qt + (1 - q)(1 - t)$.

Setting $qt + (1 - q)(1 - t) = \tfrac{1}{2} \Leftrightarrow (2q - 1)(2t - 1) = 0$, and since q is not identically equal to $\tfrac{1}{2}$, $t = \tfrac{1}{2}$.

13 (i) p(B in bag P) = p(B not chosen draw 1) + p(B chosen draw 1 and draw 2)

$$= \left(1 - \frac{k}{n}\right) + \frac{k}{n} \times \frac{k}{n + k}$$

$$= \frac{1}{n(n + k)}\left((n - k)(n + k) + k^2\right)$$

$$= \frac{n}{n + k}$$

This has its maximum value of 1 for $k = 0$, and for no other values of k. Since

$p = 1 - \dfrac{k}{n + k} \le 1$ and for $k = 0, p = 1$ but $k > 0$ for all $p < 1$).

(ii) p(Bs in same bag) = p(B_1 chosen on D_1 and neither chosen on D_2)
+ p(B_1 chosen on D_1 and both chosen on D_2)
+ p(B_1 not chosen on D_1 and B_2 chosen on D_2)

$$= \frac{k}{n} \times \frac{{}^{n+k-2}C_k}{{}^{n+k}C_k} + \frac{k}{n} \times \frac{{}^{n+k-2}C_{k-2}}{{}^{n+k}C_k} + \left(1 - \frac{k}{n}\right) \times \frac{k}{n+k}$$

Notice that, although the ${}^nC_r$ terms *look* very clumsy, they are actually quite simple once all the cancelling of common factors has been undertaken.

$$= \frac{k}{n} \times \frac{n(n-1)}{(n+k)(n+k-1)} + \frac{k}{n} \times \frac{k(k-1)}{(n+k)(n+k-1)} + \frac{k(n-k)}{n(n+k)}$$

$$= \frac{k}{n}\left\{\frac{n^2 - n + k^2 - k + (n^2 + nk - n - nk - k^2 + k)}{(n+k)(n+k-1)}\right\}$$

$$= \frac{2k(n-1)}{(n+k)(n+k-1)}$$

Differentiating this expression gives

$$\frac{\mathrm{d}p}{\mathrm{d}k} = \frac{(n^2 + 2nk + k^2 - n - k) \times 2(n-1) - 2k(n-1) \times (2n + 2k - 1)}{\left[(n+k)(n+k-1)\right]^2}$$

$= 0$ when $n^2 + 2nk + k^2 - n - k = 2nk + 2k^2 - k$ since $n > 2$, $n - 1 \neq 0$

$\Rightarrow k^2 = n(n-1)$.

Now there is nothing that guarantees that k is going to be an integer (quite the contrary, in fact), so we should look to the integers either side of the (positive) square root of $n(n-1)$:

$$k = \left\lfloor \sqrt{n(n-1)} \right\rfloor \text{ and } k = \left\lfloor \sqrt{n(n-1)} \right\rfloor + 1 .$$

In fact, since $n^2 - n = (n - ½)^2 - ¼$, $\left\lfloor \sqrt{n^2 - n} \right\rfloor = n - 1$ and we find that,

when $k = n - 1$, $p = \dfrac{2(n-1)^2}{(2n-1)2(n-1)} = \dfrac{n-1}{2n-1}$

and when $k = n$, $p = \dfrac{2n(n-1)}{(2n)(2n-1)} = \dfrac{n-1}{2n-1}$ also,

and $k = n - 1$, n are the two values required.